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BIOGRAPHY.

JAMES JOSEPH SYLVESTER, LL. D., F. R. S.

BY GEORGE BRUCE HALSTED.

ON Monday, March 15, 1897, in London, where, September 3, 1814, he was born, died the most extraordinary personage for half a century in the mathematical world.

James Joseph Sylvester was second wrangler at Cambridge in 1837. When we recall that Sylvester, Wm. Thomson, Maxwell, Clifford, J. J. Thomson were all second wranglers, we involuntarily wonder if any senior wrangler except Cayley can be ranked with them.

Yet it was characteristic of Sylvester that not to have been first was always bitter to him.

The man who beat him, Wm. N. Griffin, also a Johnion, afterwards a modest clergyman, was tremendously impressed by Sylvester, and honored him in a treatise on optics where he used Sylvester's first published paper, "Analytical development of Fresnel's optical theory of crystals," *Philosophical Magazine*, 1837.

Sylvester could not be equally generous, and explicitly rated above Griffin the fourth wrangler George Green, justly celebrated, who died in 1841.

Sylvester's second paper, "On the motion and rest of fluids," *Philosophical Magazine*, 1838 and 1839, also seemed to point to physics.

In 1838 he succeeded the Rev. Wm. Ritchie as professor of natural philosophy in University College, London.

His unwillingness to submit to the religious tests then enforced at Cambridge and to sign the 39 articles not only debarred him from his degree and from competing for the Smith's prizes, but, what was far worse, deprived him of the Fellowship morally his due. He keenly felt the injustice.

In his celebrated address at the Johns Hopkins University his denunciation of the narrowness, bigotry and intense selfishness exhibited in these compulsory creed tests, made a wonderful burst of oratory. These opinions were fully shared by De Morgan, his colleague at University College. Copies I possess of the five examination papers set by Sylvester at the June examination, session of 1839-40, show him striving as a physicist, but it was all a false start. Even his first paper shows he was always the Sylvester we knew. To the "Index of Contents" he appends the characteristic note: "Since writing this index I have made many additions more interesting than any of the propositions here cited, which will appear toward the conclusion." Ever he is borne along helpless but ecstatic in the ungovernable flood of his thought.

A physical experiment never suggests itself to the great mental experimenter. Cayley once asked for his box of drawiug instruments. Sylvester answered, "I never had one." Something of this irksomeness of the outside world, the world of matter, may have made him accept, in 1841, the professorship offered him in the University of Virginia.

On his way to America he visited Rowan Hamilton at Dublin in that observatory where the maker of quaternions was as out of place as Sylvester himself would have been. The Virginians so utterly failed to understand Sylvester, his character, his aspirations, his powers, that the Rev. Dr. Dabney, of Virginia, has seriously assured me that Sylvester was actually deficient in intellect, a sort of semi-idiotic calculating boy. For the sake of the contrast, and to show the sort of civilization in which this genius had risked himself, two letters from Sylvester's tutors at Cambridge may here be of interest.

The great Colenso, Bishop of Natal, previously Fellow and Tutor of St. John's College, writes: "Having been informed that my friend and former pupil, Mr. J. J. Sylvester, is a candidate for the office of professor of mathematics, I beg to state my high opinion of his character both as a mathematician and a gentleman.

"On the former point, indeed, his degree of Second Wrangler at the University of Cambridge would be, in itself, a sufficient testimonial. But I beg to add that his powers are of a far higher order than even that degree would certify."

Philip Kelland, himself a Senior Wrangler, and then professor of mathematics in the University of Edinburgh, writes: "I have been requested to express my opinion of the qualifications of Mr. J. J. Sylvester, as a mathematician.

"Mr. Sylvester was one of my private pupils in the University of Cambridge, where he took the degree of Second Wrangler. My opinion of Mr. Sylvester then was that in originality of thought and acuteness of perception he had never been surpassed, and I predicted for him an eminent position among the mathematicians of Europe. My anticipations have been verified. Mr. Sylves-

ter's published papers manifest a depth and originality which entitles them to the high position they occupy in the field of scientific discovery. They prove him to be a man able to grapple with the most difficult mathematical questions and are satisfactory evidence of the extent of his attainments and the vigor of his mental powers."

The five papers produced in this year, 1841, before Sylvester's departure for Virginia, show that now his key note is really struck. They adumbrate some of his greatest discoveries.

They are: "On the relation of Sturm's auxiliary functions to the roots of an algebraic equation," British Assoc. Rep. (pt. 2), 1841; "Examples of the dialytic method of elimination as applied to ternary systems of equations," Camb. M. Jour. II., 1841; On the amount and distribution of multiplicity in an algebraic equation," Phil. Mag. XVII., 1841; "On a new and more general theory of multiple roots," Phil. Mag. XVIII., 1841; "On a linear method of eliminating between double, treble and other systems of algebraic equations," Phil. Mag. XVIII., 1841; "On the dialytic method of elimination," Phil. Mag. XXI., Irish Acad. Proc. II.

This was left behind in Ireland, on the way to Virginia. Then suddenly occurs a complete stoppage in this wonderful productivity. Not one paper, not one word, is dated from the University of Virginia. Not until 1844 does the wounded bird begin again feebly to chirp, and indeed it is a whole decade before the song pours forth again with mellow vigor that wins a waiting world.

Disheartening was the whole experience; but the final cause of his sudden abandonment of the University of Virginia I gave in an address entitled, "Original Research and Creative Authorship the Essence of University Teaching," printed in *Science*, N. S., Vol. I., pp. 203-7, February 22, 1895.

On the return to England with heavy heart and dampened ardor, he takes up for his support the work of an actuary and then begins the study of law. In 1847 we find him at 26 Lincoln's Inn Fields, "eating his terms." On November 22, 1850, he is called to the bar and practices conveyancing.

But already in his paper dated August 12, 1850, we meet the significant names Boole, Cayley, and harvest is at hand.

The very words which must now be used to say what had already happened and what was now to happen were not then in existence. They were afterward made by Sylvester and constitute in themselves a tremendous contribution. As he himself says: "Names are, of course, all important to the progress of thought, and the invention of a really good name, of which the want, not previously perceived, is recognized, when supplied, as having ought to be felt, is entitled to rank on a level in importance, with the discovery of a new scientific theory."

Elsewhere he says of himself: "Perhaps I may without immodesty lay claim to the appellation of the Mathematical Adam, as I believe that I have given more names (passed into general circulation) to the creatures of the mathematical reason than all the other mathematicians of the age combined."

In one year, 1851, Sylvester created a whole new continent, a new world in the universe of mathematics. Demonstration of its creation is given by the Glossary of New Terms which he gives in the *Philosophical Transactions*, Vol. 143, pp. 543-548.

Says Dr. W. Franz Meyer in his exceedingly valuable Bericht über die Fortschritte der projectiven Invariantentheorie, the best history of the subject (1892):

"Als äusseres Zeichen für den Umfang der vorgeschrittenen Entwickelung mag die ausgedehnte, grösstenteils von *Sylvester* selbst herrührende Terminologie dienen, die sich am Ende seiner grossen Abhandlung über Sturm'sche Functionen (1853) zusammengestellt findet."

Using then this new language, let us briefly say what had happened in the decade when Sylvester's genius was suffering from its Virginia wound. The birth-day of the giant *Theory of Invariants* is April 28, 1841, the date attached by George Boole to a paper in the Cambridge *Mathematical Journal* where he not only proved the invariantive property of discriminants generally, but also gave a simple principle to form simultaneous invariants of a system of two functions. The paper appeared in November, 1841, and shortly after, in February, 1842, Boole showed that the polars of a form lead to a broad class of covariants. Here he extended the results of the first article to more than two Forms. Boole's papers led Cayley, nearly three years later (1845), to propose to himself the problem to determine *a priori* what functions of the coefficients of an equation possess this property of invariance, and he discovered its possession by other functions besides discriminants, for example the quadrinvariants of binary quantics, and in particular the invariant S of a quartic.

Boole next discovered the other invariant T of a quartic and the expression of the discriminant in terms of S and T. Cayley next (1846) published a symbolic method of finding invariants. Early in 1851 Boole reproduced, with additions, his paper on Linear Transformations; then at last began Sylvester. He always mourned what he called "the years he lost fighting the world"; but, after all, it was he who made the Theory of Invariants.

Says Meyer: "sehen wir in dem Cyklus *Sylvester*'scher Publicationen (1851–1854) bereits die Grundzüge einer allgemeinen Theorie erstehen, welche die Elemente von den verschiedenartigsten Zweigen der späteren Disciplin umfasst." "Sylvester beginnt damit, die Ergebnisse seiner Vorgänger unter einem einzigen Gesichtspunkte zuvereinigen."

With deepest foresight Sylvester introduced, together with the original variables, those dual to them, and created the theory of contravariants and intermediate forms. He introduced, with many other processes for producing invariantive forms, the principle of mutual differentiation.

Hilbert attributes the sudden growth of the theory to these processes for producing and handling invariantive creatures. "Die Theorie dieser Gebilde erhob sich, von speciellen Aufgaben ausgehend, rasch zu grosser Allgemeinheit —dank vor Allem dem Umstande, dass es gelang, eine Reihe von besonderen der Invariantentheorie eigenthümlichen Prozessen zu entdecken, deren An-

wendung die Aufstellung und Behandlung invarianter Bildungen beträchtlich erleichterte."

"Was die Theorie der algebraischen Invarianten anbetrifft so sind die ersten Begründer derselben, *Cayley* und *Sylvester*, zugleich auch als die Vertreter der naiven Periode anzusehen: an der Aufstellung der einfachsten Invariantenbildungen und an den eleganten Anwendungen auf die Auflösung der Gleichungen der ersten 4 Grade hatten sie die unmittelbare Freude der ersten Entdeckung." It was Sylvester alone who created the theory of canonic forms and proceeded to apply it with astonishing power. What marvelous mass of brand new being he now brought forth!

Moreover he trumpeted abroad the eruption. He called for communications to himself in English, French, Italian, Latin or German, so only the "Latin character" were used.

From 1851 to 1854 he produces forty-six different memoirs. Then comes a dead silence of a whole year, broken in 1856 by a feeble chirp called "A Trifle on Projectiles."

What has happened? Some more "fighting the world." Sylvester declared himself a candidate for the vacant professorship of geometry in Gresham College, delivered a probationary lecture on the 4th of December, 1854, and was ignominiously "turned down." Let us save a couple of sentences from this lecture:

"He who would know what geometry is must venture boldly into its depths and learn to think and feel as a geometer. I believe that it is impossible to do this, to study geometry as it admits of being studied, and I am conscious it can be taught, without finding the reasoning invigorated, the invention quickened, the sentiment of the orderly and beautiful awakeded and enhanced, and reverence for truth, the foundation of all integrity of character, converted into a fixed principle of the mental and moral constitution, according to the old and expressive adage '*abeunt studia in mores.*'"

But this silent year concealed still another stunning blow of precisely the same sort, as bears witness the following letter from Lord Brougham to The Lord Panmure:

"BROUGHAM,

PRIVATE. 28 Aug. 1855.

MY DEAR P.

My learned excellent friend and brother mathematician Mr. Sylvester is again a candidate for the professorship at Woolwich on the death of Mr. O'Brian who carried it against him last year.

I entreat once more your favorable consideration of this eminent man who has already to thank you for your great kindness.

Yours sincerely,

H. BROUGHAM.

On his third trial, backed by such an array of credentials as no man ever presented before, he barely scraped through, was appointed professor of mathematics at the Royal Military Academy, and served at Woolwich exactly 14 years, 10 months, and 15 days.

A single sentence of his will best express his greatest achievement there and his manner of exit thence:

"If Her most Gracious Majesty should ever be moved to recognize the palmary exploit of the writer of this note in the field of English science as having been the one successfully to resolve a question and conquer an algebraical difficulty which had exercised in vain for two centuries past, since the time of Newton, the highest mathematical intellects in Europe (Euler, Lagrange, Maclaurin, Waring among the number), by conferring upon him some honorary distinction in commemoration of the deed, he will crave the privilege of being allowed to enter the royal presence, not covered, like De Courcy, but barefooted, with rope around his waist, and a *goose*-quill behind his ear, in token of repentant humility, and as an emblem of convicted simplicity in having once supposed that on such kind of success he could found any additional title to receive fair and just consideration at the hands of Her Majesty's Government when quitting his appointment as public professor at Woolwich under the coercive operation of a non-Parliamentary retrospective and utterly unprecedented War Office enactment." Athenæum Club, January 31, 1871. Of course this means a row of barren years, 1870, 1871, 1872, 1873.

The fortunate accident of a visit paid Sylvester in the autumn of 1873 by Pafnuti Lvovich Chebyshev, of the Universty of St. Petersburg, reawakened our genius to produce in a single burst of enthusiasm a new branch of science.

On Friday evening, January 23, 1874, Sylvester delivered at the Royal Institution a lecture entitled "On Recent Discoveries in Mechanical Conversion of Motion," whose ideas, carried on by two of his hearers, H. Hart and A. B. Kempe, have made themselves a permanent place even in the elements of geometry and kinematics. A synopsis of this lecture was published, but so curtailed and twisted into the third person that the life and flavor are quite gone from it. I possess the unique manuscript of this epoch-making lecture as actually delivered. A few sentences will show how characteristic and inimitable was the original form:

"The air of Russia seems no less favorable to mathematical acumen than to a genius for fable and song. Lobacheffsky, the first to mitigate the severity of the Euclidean code and to beat down the bars of a supposed adamantine necessity, was born (a Russian of Russians), in the government of Nijni Novgorod; Tchebicheff [Chebyshev], the prince and conqueror of prime numbers, able to cope with their refractory character and to confine the stream of their erratic flow, their progression, within algebraic limits, in the adjacent circumscription of Moscow; and our own Cayley was cradled amidst the snows of St. Petersburg." [Sylvester himself contracted Chebyshev's limits for the distribution of primes.] "I think I may fairly affirm that a simple direct solution of the problem of the duplication of the cube by mechanical means was never accomplished down to this day. I will not say but that, by a merciful interpretation of his oracle, Apollo may have put up with the solution which the ancient geometers obtained by means of drawing two parabolic curves; but of this I feel assured that had I been

then alive, and could have shown my solution, which I am about to exhibit to you, Apollo would have leaped for joy and danced (like David before the ark), with my triple cell in hand, in place of his lyre, before his own duplicated altar."

That in the very next year Sylvester was taking a more active part than has hitherto been known in the organization of the incipient Johns Hopkins University is seen from the following letter to him in London from the great Joseph Henry:

SMITHSONIAN INSTITUTION,
August 25, 1875.

MY DEAR SIR:

Your letter of the 13th inst. has just been received and in reply I have to say that I have written to President Gilman of the Hopkins University giving my views as to what it ought to be and have stated that if properly managed it may do more for the advance of literature and science in this country than any other institution ever established; it is entirely independent of public favor and may lead instead of following popular opinion.

I have advised that liberal salaries be paid to the occupants of the principal chairs and that to fill them the best men in the world who can be obtained should be secured.

I have mentioned your name prominently as one of the very first mathematicians of the day; what the result will be, however, I can not say.

The Trustees are all citizens of Baltimore and among them I have some personal friends; the President, Mr. Gilman, and one of them, came to Washington a few weeks ago to get from me any suggestions that I might have to offer.

It is to be regretted that in this country the Trustees, who control the management of bequests of this character, think it important to produce a palpable manifestation of the institution to be established by spending a large amount of the bequest in architectural displays. Against this custom I have protested and have asserted that if the proper men and the necessary implements of instruction are provided, the teaching may be done in log cabins.

It would give me great pleasure to have you again as my guest, and I will do what I can to secure your election. Very truly your friend,

JOSEPH HENRY.

We know the result.

Sylvester was offered the place; demanded a higher salary; won; came.

I was his first pupil, his first class, and he always insisted that it was I who brought him back to the Theory of Invariative Forms. In a letter to me of September 24, 1882, he writes: "Nor can I ever be oblivious of the advantage which I derived from your well-grounded persistence in inducing me to lecture on the Modern Algebra, which had the effect of bringing my mind back to this subject, from which it had for some time previously been withdrawn, and in which I have been laboring, with a success which has considerably exceeded my anticipations, ever since."

He made this same statement at greater length in his celebrated address at the Johns Hopkins on February 22, 1877: "At this moment I happen to be engaged in a research of fascinating interest to myself, and which, if the day only responds to the promise of its dawn, will meet, I believe, a sympathetic response from the professors of our divine algebraical art wherever scattered through the world.

"There are things called Algebraical Forms; Professor Cayley calls them

Quantics. These are not, properly speaking, Geometrical Forms, although capable, to some extent, of being embodied in them, but rather schemes of processes, or of operations for forming, for calling into existence, as it were, algebraic quantities.

"To every such Quantic is associated an infinite variety of other forms that may be regarded as engendered from and floating, like an atmosphere, around it; but infinite in number as are these derived existences, these emanations from the parent form, it is found that they admit of being obtained by composition, by mixture, so to say, of a certain limited number of fundamental forms, standard rays, as they might be termed, in the Algebraic Spectrum of the Quantic to which they belong; and, as it is a leading pursuit of the physicists of the present day to ascertain the fixed lines in the spectrum of every chemical substance, so it is the aim and object of a great school of mathematicians to make out the fundamental derived forms, the Covariants and Invariants, as they are called, of these Quantics.

"This is the kind of investigation in which I have, for the last month or two, been immersed, and which I entertain great hopes of bringing to a successful issue.

"Why do I mention it here? It is to illustrate my opinion as to the invaluable aid of teaching to the teacher, in throwing him back upon his own thoughts and leading him to evolve new results from ideas that would have otherwise remained passive or dormant in his mind.

"But for the persistence of a student of this university in urging upon me his desire to study with me the modern algebra I should never have been led into this investigation; and the new facts and principles which I have discovered in regard to it (important facts, I believe) would, so far as I am concerned, have remained still hidden in the womb of time. In vain I represented to this inquisitive student that he would do better to take up some other subject lying less off the beaten track of study, such as the higher parts of the Calculus or Elliptic Functions, or the theory of Substitutions, or I wot not what besides. He stuck with perfect respectfulness, but with invincible pertinacity, to his point. He would have the New Algebra (Heaven knows where he had heard about it, for it is almost unknown on this continent), that or nothing. I was obliged to yield, and what was the consequence? In trying to throw light upon an obscure explanation in our text-book my brain took fire; I plunged with requickened zeal into a subject which I had for years abandoned, and found food for thoughts which have engaged my attention for a considerable time past, and will probably occupy all my powers of contemplation advantageously for several months to come."

Another specific instance of the same thing he mentions in his paper, "Proof of the Hitherto Undemonstrated Fundamental Theorem of Invariants," dated November 13, 1877:

"I am about to demonstrate a theorem which has been waiting proof for the last quarter of a century and upwards. It is the more necessary that this

should be done, because the theorem has been supposed to lead to false conclusions, and its correctness has consequently been impugned. Thus in Professor Faà de Bruno's valuable *Théorie des formes binaires*, Turin, 1876, at the foot of page 150 occurs the following passage: "Cela suppose essentiellement que les équations de condition *soient toutes indépendantes entr'elles, ce qui n'est pas toujours le cas*, ainsi qu'il résulte des recherches du Professor Gordan sur les nombres des covariants des formes quintique et sextique."

The reader is cautioned against supposing that the consequence alleged above does result from Gordan's researches, which are indubitably correct. This supposed consequence must have arisen from a misapprehension, on the part of M. de Bruno, of the nature of Professor Cayley's rectification of the error of reasoning contained in his second memoir on Quantics, which had led to results discordant with Gordan's. Thus error breeds error, unless and until the pernicious brood is stamped out for good and all under the iron heel of rigid demonstration. In the early part of this year Mr. Halsted, a fellow of Johns Hopkins University, called my attention to this passage in M. de Bruno's book; and all I could say in reply was that 'the extrinsic evidence in support of the independence of the equations which had been impugned rendered it in my mind as certain as any fact in nature could be, but that to reduce it to an exact demonstration transcended, I thought, the powers of the human understanding.' "

In 1883 Sylvester was made Savilian professor of geometry at Oxford, the first Cambridge man so honored since the appointment of Wallis in 1649.

To greet the new environment, he created a new subject for his researches —Reciprocants, which has inspired, among others, J. Hammond, of Oxford; McMahon, of Woolwich; A. R. Forsyth, of Cambridge; Leudesdorf, Elliott and Halphen.

Sylvester never solved exercise problems such as are proposed in the *Educational Times*, though he made them all his life long down to his latest years. For example, *unsolved* problems by him will be found even in Vol. LXII. and Vol. LXIII. of the *Educational Times* reprints (1895). If at the time of meeting his own problem he met also a neat solution he would communicate them together, but he never solved any. In the meagre notices that have been given of Sylvester the strangest errors abound. Thus C. S. Pierce, in the *Post*, March 16th, speaks of his accepting, "with much diffidence," a word whose meaning he never knew; and gives 1862 as the date of his retirement from Woolwich, which is eight years wrong, as this forced retirement was July 31, 1870, after his 55th birthday. Cajori, in his inadequate account (History of Mathematics, p. 326), puts the studying of law before the professorship at University College and the professorship at the University of Virginia, both of which it followed. Effect must follow cause. And strange, that of the few things he ascribes to Sylvester, he should have hit upon something not his, "the discovery of the partial differential equations satisfied by the invariants and covariants of binary quantics." But Sylvester has explicitly said in Section VI. of his "Calculus of Forms:" "I alluded to the partial differential equations by which every invariant may be de-

fined. M. Aronhold, as I collect from private information, was the first to think of the application of this method to the subject; but it was Mr. Cayley who communicated to me the equations which define the invariants of functions of two variables."

Surely he needs nothing but his very own, this marvellous man who gave so lavishly to every one devoted to mathematics, or, indeed, to the highest advance of human thought in any form.

University of Texas.

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from March Number.]

III. Proofs Resulting from Comparison of Areas.

Note. Under this head, only a few varieties in connection with each type of figure will be given. The possible number of varieties of "dissection proofs" is absolutely unlimited.

XXXIII. Fig. 25.

Rectangle AM is equivalent to $2\triangle FAC$ is equivalent to $2\triangle EAB$ is equivalent to square EC. Similarly, rectangle BM is equivalent to square KC.

$\therefore$ Adding, square AH is equivalent to square EC+square KC.

Euclid's Proof. Prop. 47, Book I.

XXXIV. Fig. 25.

Rectangle AM is equivalent to parallelogram Aa=parallelogram AO is equivalent to square AD. Similarly, rectangle BM is equivalent to square BL.

$\therefore$ Adding, square AH is equivalent to square AD+square BL.

Edwards's Geometry, page 160.

XXXV. Fig. 25.

$AB\cdot Ad$ is equivalent to $ABOE$ is equivalent to $ACDE$.

$AB\cdot Be$ is equivalent to $ABKP$ is equivalent to $BKLC$.

$\therefore$ Adding, $AB(Ad+Be)$ is equivalent to $AB(Ad+dR)$ is equivalent to $AB\cdot AF=ABHF$ is equivalent to $ACDE+BKLC$.

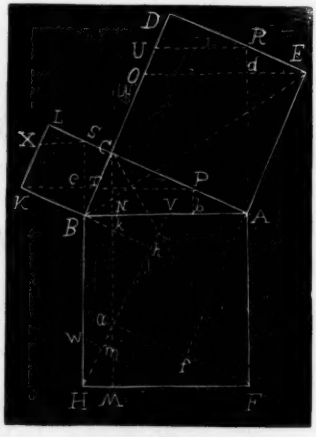


Fig. 25.

XXXVI. Fig. 25.

AFMN is equivalent to *AFaC* is equivalent to $AC \cdot Af = ACDE$. Similarly, *BHMN* is equivalent to *BKLC*.

$\therefore$ Adding, *ABHF* is equivalent to $ACDE + BKLC$.

Vieth, 1805.

XXXVII. Fig. 25.

$FMa = AdE$; $AfF = EDO$; $ANaf = AdOc$.

$\therefore$ *AFMN* is equivalent to *ACDE*. Similarly, *BHMN* is equivalent to *BKLC*.

$\therefore$ *ABHF* is equivalent to $ACDE + BKLC$.

E. von Littrow, 1839.

XXXVIII. Fig. 25.

$AVaF = RUCA$; $FaH = AER$; $HhB = PLK$ is equivalent to $RDU + CLKT$; $BhV = KBT$.

$\therefore$ *ABHF* is equivalent to $ACDE + BKLC$.

XXIX. Fig. 25.

$AVaF = RUCA$; $FaH = AER$; $RDU = HmW$; $BhmW = BKLS$; $BhV = BCS$.

$\therefore$ *ABHF* is equivalent to $ACDE + BKLC$.

XL. Fig. 26.

AFMR is equivalent to *ACNO* is equivalent to *ACDE*. So, *BHMR* is equivalent to *BKLC*.

$\therefore$ *ABHF* is equivalent to $ACDE + BKLC$.

Sechhio, 1753.

XLI. Fig. 26.

AFMR is equivalent to $AFaC = ESLD$ is equivalent to *ACDE*. So, *BHMR* is equivalent to *BKLC*.

$\therefore$ *ABHF* is equivalent to $ACDE + BKLC$.

Edwards's Geometry, page 158.

XLII. Fig. 26.

$CAFU = ACNE$, and $CaU = CND$.

$\therefore$ *FaCA* (is equivalent to *AFMR*) is equivalent to *ACDE*. So, *BHMR* is equivalent to *BKLC*.

$\therefore$ *ABHF* is equivalent to $ACDE + BKLC$.

XLIII. Fig. 26.

$Fa = AC = Va$.

$\therefore$ $Fa \cdot Va$ (is equivalent to $AFaC$) $= ACDE$.

$\therefore$ *ARMF* is equivalent to *ACDE*. So, *BHMR* is equivalent to *BKLC*.

$\therefore$ *ABHF* is equivalent to $ACDE + BKLC$.

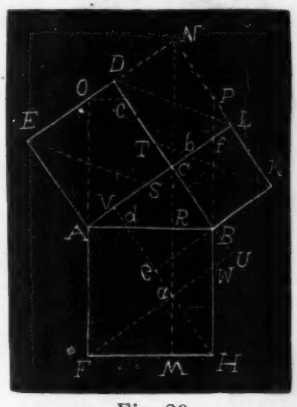


Fig. 26.

XLIV. Fig. 26.

$AFHd = OABc$ is equivalent to $AEOcC$.

$BeH = KBT$ is equivalent to $KBCb + ODc$. $Bed = KLb$.

$\therefore$ $ABHF$ is equivalent to $ACDE + BKLC$.

XLV. Fig. 26.

$HaW = KLb$. $AFHaWB$ is equivalent to $ACBWF$ is equivalent to $ONPfA$ is equivalent to $ONCA + NPfC$ is equivalent to $ACDE + BKbC$.

$\therefore$ $ABHF$ is equivalent to $ACDE + BKLC$.

[To be Continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from April Number.]

PROPOSITION XXVII. *If a straight AX (Fig. 32.) drawn at any however small angle from the point A of AB, must at length meet (anyhow at an infinite distance) any perpendicular BX, which is supposed erected at any distance from this point A upon the secant AB: I say there will then be no more place for the hypothesis of acute angle.*

PROOF. From any point K chosen at will in AB near the point A, the perpendicular KL is erected to AB, which certainly (from Cor. II. of the preceding proposition) meets AX at a finite or terminated distance in some point L. But now it holds that there may be assumed in KB portions KK each equal to a certain assignable length R, and these more than any assignable finite number; since indeed the point B can be situated, in accordance with the present supposition, at however great a distance from this point A.

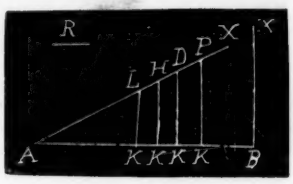


Fig. 32.

And accordingly from the other points K are erected to AB perpendiculars KH, KD, KP, which all (from the aforesaid corollary) meet the straight AX in certain points H, D, P; and so about the remaining points K uniformly designated toward the point B.

It holds secondly (from Eu. I. 16) that the angles at the points L, H, D, P will all be obtuse toward the parts of the points X; and just so (from Eu. I. 13) the angles at the aforesaid points will all be acute toward the point A.

Therefore (from Cor. II. after 3 of this) the side KH will be greater than the side KL ; the side KD greater than the side KH ; and so always proceeding towards the points X.

It holds thirdly that the four angles together of the quadrilateral $KLHK$ will be greater than the four angles together of the quadrilateral $KHDK$: for this in like case has already been demonstrated in XXIV of this.

It holds fourthly that the same is valid likewise of the quadrilateral $KHDK$ in relation to the quadrilateral $KDPK$; and so on always, proceeding to quadrilaterals more remote from this point A.

Since therefore are present (as in XXV of this) as many quadrilaterals described in the aforesaid mode, as there are, except the first LK, perpendiculars let fall from points of AX to the straight AB, it will hold uniformly (if we assume nine perpendiculars of this sort let fall, besides the first) the sum of all the angles which are comprehended by these nine quadrilaterals will exceed 35 right angles ; and therefore the four angles together of the first quadrilateral $KLHK$, which indeed in this regard has been shown the greatest of all, will fall short of four right angles by less than the ninth part of one right angle. Wherefore, these quadrilaterals being multiplied beyond any assignable finite number, proceeding always toward the parts of the points X, it holds in the same way (as in the same already recited theorem) that the four angles together of this stable quadrilateral $KHLK$ will fall short of four right angles less than any assignable little portion of one right angle.

Therefore these four angles together will be either equal to four right angles, or greater.

But then (from XVI of this) is established the hypothesis either of right angle or of obtuse angle ; and therefore (from V and VI of this) is destroyed the hypothesis of acute angle.

So then it holds, that there will be no place for the hypothesis of acute angle, if the straight AX drawn under however small angle from the point A of AB must at length meet (anyhow at an infinite distance) any perpendicular BX, which is supposed erected at any distance from this point A upon this secant AB.

Quod erat etc.

SOME DIVISIBILITY TESTS.

By WM. E. HEAL, Member of the London Mathematical Society, Marion, Indiana.

In the *Educational Times* for March, 1897, Professor Sylvester proposed the following problem : "If the digits r in number of any integer N read from left to right be multiplied repeatedly by the first r terms of the recurring series

1, 4, 3, −1, −4, −3 ; $\dot{1}$, $\dot{4}$, $\dot{3}$, $-\dot{1}$, $-\dot{4}$, $-\dot{3}$, show that, if the sum of these products be divisible by 13, so N will be, and not otherwise." The reason for the rule is apparent when we notice that 1, 4, 3, −1, −4, −3 are the remainders in reverse order of 10^1, 10^2, 10^3, 10^4, 10^5, 10^6 mod. 13 ; or what is the same thing in the development of $\frac{1}{13}$ as a circulating decimal.

Since we may prefix any number of ciphers to any number, it is clear that we may start with any number of the series only being careful to preserve the cyclical order. For example, we might equally as well write the series 3, −1, −4, −3, 1, 4.

Example. 11140640173 is divisible by 13 because $1(1)+4(1)+3(1)-1(4)-4(0)-3(6)+1(4)+4(0)+3(1)-1(7)-4(3)=-26=-2(13)$.

728 is divisible by (13) because $3(7)-1(2)-4(8)=-13$.

The reason for the rule suggests its extension to any number whatever.

Thus $\frac{1}{3}$ developed in a circulating decimal gives the constant remainder 1 and we have the well known rule that a number is divisible by 3 if the sum of its digits is so. $\frac{1}{7}$ developed in a circulating decimal gives the series 2, 3, 1, −2, −3, −1. Thus 6028620892 is divisible by 7 because $2(6)+3(0)+1(2)-2(8)-3(6)-1(2)+2(0)+3(8)+1(9)-2(2)=7$.

For 11 the remainders are 1, −1, and we have the known rule for divisibility by 11. For 13 the rule is as stated by Sylvester. For 17 we find the series 1, −5, 8, −6, −4, 3, 2, 7, −1, 5, −8, 6, 4, −3, −2, −7. Thus 442 is divisible by 17 because $3(4)+2(4)+7(2)=34=2(17)$.

For 19 we have the series 1, 2, 4, 8, −3, −6, 7, −5, 9, −1, −2, −4, −8, 3, 6, −7, 5, −9. It is clear that in this way we can find similar tests of divisibility for any number whatever, but it does not seem worth while to push the matter further except in special cases.

A simple rule for divisibility by 37 may be found in this way. The remainders are 1, −11, 10. Thus 343619 is divisible by 37 because $1(3)-11(4)+10(3)+1(6)-11(1)+10(9)=74=2(37)$.

May 7, 1897.

INTRODUCTION TO DIFFERENTIATION.

By JOHN MACNIE, A. M., Professor of Mathematics, University of North Dakota.

1. In the identity $\frac{r^n-1}{r-1}=r^{n-1}+r^{n-2}+\ldots\ldots r+1$, ... (1),

Since r may have any value, let $r=\frac{x^{1/m}}{z^{1/m}}$; then, by substituting this value for r in (1), multiplying both members by $z^{\frac{n-1}{m}}$, and simplifying, we obtain

$$\frac{x^{n/m}-z^{n/m}}{x^{1/m}-z^{1/m}} = x^{\frac{n-1}{m}} + x^{\frac{n-2}{m}}z^{\frac{1}{m}} + \ldots\ldots x^{\frac{1}{m}}z^{\frac{n-2}{m}} + z^{\frac{n-1}{m}} \quad \ldots\ldots\ldots(2).$$

Dividing both members of (2) by the factor that rendered $x^{1/m}-z^{1/m}$ rational, we obtain, since, by (2), $x-z=(x^{1/m}-z^{1/m})(x^{\frac{n-1}{m}}+\ldots\ldots z^{\frac{n-1}{m}})$,

$$\frac{x^{n/m}-z^{n/m}}{x-z} = \frac{x^{\frac{n-1}{m}}+z^{\frac{n-2}{m}}z^{\frac{1}{m}}+\ldots\ldots x^{\frac{1}{m}}z^{\frac{n-2}{m}}+z^{\frac{n-1}{m}}}{x^{\frac{m-1}{m}}+x^{\frac{m-2}{m}}z^{\frac{1}{m}}+\ldots\ldots x^{\frac{1}{m}}z^{\frac{m-2}{m}}+z^{\frac{n-1}{m}}} \quad \ldots\ldots\ldots(3),$$

which, as m may have any value, ± 1 included, is a general expression for the *ratio of the difference of two like powers to the difference of their bases.*

In (3), if we suppose $z=x$, since, then, there are in the numerator of the second member n terms, each $=x^{\frac{n-1}{m}}$, and in the denominator m terms, each $=x^{\frac{m-1}{m}}$ we obtain, for $z=x$,

$$\left[\frac{x^{\frac{n}{m}}-z^{\frac{n}{m}}}{x-z}\right]_{x=z} = \frac{0}{0} = \frac{nx^{\frac{n-1}{m}}}{mx^{\frac{m-1}{m}}} = \frac{n}{m}x^{(n/m)-1}$$

the first member assuming the indeterminate form on account of the presence in numerator and denominator of the factor $x^{1/m}-z^{1/m}$, which becomes zero by hypothesis. Hence, as m may have any value, the formula

$$\left[\frac{x^n-z^n}{x-z}\right]_{x=z} = nx^{n-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

holds true for every value of n. For the sake of simplicity of statement we shall suppose in what immediately follows $m=1$, and $n=$a positive integer.

Then (3) becomes

$$\frac{x^n-z^n}{x-z} = x^{n-1}+x^{n-2}z+\ldots\ldots xz^{n-2}+z^{n-1} \quad \ldots\ldots\ldots\ldots(3').$$

2. Now, instead of regarding x and z in (3') as unknown constants, we may regard them as denoting different values of the same variable z, as it varies from $z=0$, through $z=x$, toward $z=+\infty$. From this point of view we see that, assigning any two values to x and z, each member of (3') expresses *the ratio of the increment of the power to the increment of the base, between these values;* or, briefly expressed, gives the *rate of increase of* z^n. For example, let $z=0$, $x=a$; then both members of (3') become a^{n-1}, the *average* rate of increase of z^n while z increases from 0 to a; i. e. while z has increased by a units, z^n has increased a^{n-1} times as fast. We say "average rate" because, as will be seen by giving different values to a, the rate of increase of z^n is continually accel-

erating, just as the velocity or rate of motion of a falling body is continually accelerating.

3. If now we suppose $z=x-h$, h being infinitely small, the second member of (3′) will be *less* than nx^{n-1} by a difference infinitely small ; and if we suppose $z=x+h$, the second member of (3′) will be *greater* than nx^{n-1} by a difference infinitely small ; we infer, accordingly (the values of that second member being continuous) that nx^{n-1} represents the *rate of increase of* z^n *when* z *is passing through* x. For, if nx^{n-1} does not represent the rate of increase of z^n when z is passing through x, for what value of z does it represent the rate ?

The difficulty that is here experienced arises from the fact that we have here to deal, not with a constant ratio, as in algebra, but with a ratio that varies continuously as its terms vary, ratios of frequent occurence in physics and kindred sciences. Thus, when we say that a falling body at a certain point in its descent has a velocity of 50 feet a second, we do not mean that the body moves at that rate during any assignable period of time, but *would* descend that distance in a second, *if the motion continued uniform*. In the same way, nx^{n-1} does not mean the rate of increase of z^n during an interval of increase of z but the rate at which z^n would increase if the rate became constant from x.

From the limitation of our faculties, we are unable to realize the *absolute ;* as, for example, to draw or even conceive a straight line absolutely without breadth. Yet, while admitting this inability, we ignore in our reasonings upon straight lines all that is inconsistent with their definition. Similarly, while in our conception of a variable, a changing velocity for example, we can not help thinking of the *element* of change as constant for some interval, however minute, we here, again, ignore whatever is inconsistent with the definition of a variable as changing continuously. There is no objection, then, to our assisting our grasp of the idea by regarding a power of a variable as changing by infinitely small constant* *elements*, as long as we ignore inconsistent consequences.

4. DEF. *Function*, as usual. Example, x^n a function of x.

5. DEF. A variable being supposed to change by infinitely small *elements*, such an element is called the *differential* of the variable. The differential of a variable is denoted by the symbol d prefixed to the symbol of the variable. Thus dx, $d(x^n)$, are read respectively, *the differential of* x, *the differential of* x^n.

It has already been seen that $d(x^n)=nx^{n-1}dx$, that is when the variable is passing through the value x, the power is changing nx^{n-1} times as fast as the variable. Hence nx^{n-1} is called *the differential coefficient* of x^n, etc.

6. (Here would follow the demonstration of the rules for algebraic sums of variables, found much as usual. The rule for the differential of products may be found as follows, without the intervention of series.)

7. *To find the differential of the product of two variables*, say xy.

$\because\ 2xy=(x+y)^2-x^2-y^2.$

$\therefore\ 2d(xy)=2(x+y)\times d(x+y)-d(x^2)-d(y^2).$

*That is, constant during an infinitely small interval.

$$2d(xy)=2(x+y)(dx+dy)-2xdx-2ydy.$$

i. e. $d(xy)=xdx+xdy+ydx+ydy-xdx-ydy.$

i. e. $d(xy)=xdy+ydx.$

From this may be derived rules for $d(xyz)$, etc., and $d(x/y)$.

8. Here would follow demonstration by differentials of Binomial Formula for all values of n, with exercises.

9. Here would follow the algebraic deduction of some such formula as:

$$\log(1+z)=M(z-\tfrac{1}{2}z^2+\tfrac{1}{3}z^3-\ldots\ldots \text{ ad inf.})$$

$$\text{whence } d(\log 1+z)=M(1-z+z^2-\ldots\ldots\text{ad inf.})dx$$

$$d(\log 1+z)=M.\frac{1}{1+z}.dx$$

or, putting x for $1+z$ we have

$$d\log x=M(dx/x).$$

Whence may be derived $d(a^x)=a^x\log a$, etc.

10. Here would follow the algebraic deduction of

$$\sin x=x-(x^3/3\,!)+(x^5/5\,!)-(x^7/7\,!)+\ \ldots\ldots \qquad (1)$$

$$\text{and } \cos x=1-(x^2/2\,!)+(x^4/4\,!)-(x^6/6\,!)+\ldots\ldots \qquad (2).$$

From (1), $d(\sin x)=\{1-(x^2/2\,!)+(x^4/4\,!)-(x^6/6\,!)+\ \ldots\ldots\}dx=\cos x dx$,

and from (2), $d(\cos x)=-\sin x dx$, etc.

11. Then might follow applications to questions of maxima and minima, etc. Then deduction of Taylor's Theorem, with applications.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

79. Proposed by F. M. PRIEST, St. Louis, Mo.

How many $20 gold pieces can be put in a room 20 feet long, 18 feet wide, 9 feet high?

Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Spring-Springfield, Missouri.

A \$20 gold piece is about $_{1\,0\,0}^{8}$ of an inch thick, and about $1\frac{7}{10}$ inches in diameter. By putting the pieces in cylindrical layers lengthwise of the room, we can place $(18\times 12)\div 1\frac{7}{10}$ or 160 cylinders in the first layer, each cylinder containing $(20\times 12)\div {}_{1\,0\,0}^{8}$ or 3000 \$20 gold pieces. By rectangular arrangement of the cylinders we can put in $(9\times 12)\div 1\frac{7}{10}$ or 80 layers. Hence, by this arrangement, we can put $80\times 160\times 3000=38{,}400{,}000$ pieces in the room.

By laying the cylinders of the second layer of cylinders between two cylinders of the first layer, the distance between the plane of centers of the first layer and the plane of centers of the second layer is $\sqrt{(\frac{2}{2}\frac{7}{0})^2-(\frac{2}{4}\frac{7}{0})^2}=\frac{2}{4}\frac{7}{0}\sqrt{3}$. Hence, there can be placed in the room, by this arrangement, $(9\times 12)\div\frac{2}{4}\frac{7}{0}\sqrt{3}$ or 92 layers $+.376$ of a layer.

In these 92 layers 46 layers would contain 160 cylinders and 46 would cantain 159. But since there is still room at the top the last layer can be placed in so as to contain 160 cylinders.

Hence, there will be 47 layers of 160 cylinders and 45 layers of 159.

Since each cylinder contains 3000 \$20 gold pieces, there can be placed in the room by this method $(47\times 160+45\times 159)\times 3000=44{,}025{,}000$ pieces.

It is possible that by considering other dimensions in the same way as the width in this solution a still larger number may be placed in the room.

Charles C. Cross obtained as his answer 38,400,000.

80. Proposed by CHARLES C. CROSS, Laytonsville, Maryland.

From a cask containing 10 gallons of wine, a servant drew off 1 gallon each day, for five days, each time supplying the deficiency by adding a gallon of water. Afterwards, fearing detection, he again drew off a gallon a day for five days, adding each time a gallon of wine. How many gallons of water still remained in the cask? [From *Quackenbos' Arithmetic.*]

Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Let 10 gallons$=a$, 1 gallon$=b$, the quantity of water or wine added after each draught, $\frac{1}{1\,0}=b/a=1/n$, the part drawn off each time.

Then $a-a/n=a\left(\frac{n-1}{n}\right)$=quantity of wine left after first draught;

$a\left(\frac{n-1}{n}\right)-1/n$ of $a\left(\frac{n-1}{n}\right)=a\left(\frac{n-1}{n}\right)^2$=quantity of wine left after second draught;

$a\left(\frac{n-1}{n}\right)^2-1/n$ of $a\left(\frac{n-1}{n}\right)^2=a\left(\frac{n-1}{n}\right)^3$=quantity of wine left after third draught; and $a\left(\frac{n-1}{n}\right)^m$=quantity left after the mth draught$=A$.

Then $a-A$=water in the cask.

$A+b$=quantity of wine in cask before the $(m+1)$th draught since b gallons of wine are added.

$$A+b-[(A/n)+(b/n)]+b=A\left(\frac{n-1}{n}\right)+b\left(\frac{n-1}{n}\right)=\text{quantity of wine before the } (m+2)\text{th draught.}$$

$$A\left(\frac{n-1}{n}\right)+b\left(\frac{2n-1}{n}\right)-A\left(\frac{n-1}{n^2}\right)-b\left(\frac{2n-1}{n^2}\right)+b=A\left(\frac{n-1}{n}\right)^2-b\left(\frac{3n^2-3n+1}{n^2}\right)$$

=quantity of wine before the $(m+3)$th draught.

$$\therefore A\left(\frac{n-1}{n}\right)^p+b\left(pn^{p-1}-\frac{p(p-1)}{1.2}n^{p-2}+\ldots\ldots\right)+b$$

$$=A\left(\frac{n-1}{n}\right)^p+b\left(\frac{n^p-(n-1)^p}{n^p}\right)$$

$$=\text{quantity of wine left after } (m+p)\text{th draught}=a\left(\frac{n-1}{n}\right)^{m+p}+b\left(\frac{n^p-(n-1)^p}{n^p}\right)$$

In the present case, $a=10$, $b=1$, $1/m=\frac{1}{10}$, $m=5$, and $p=5$. Hence, substituting, we have $10\left[\frac{10-1}{10}\right]^{10}+1.\left[\frac{10^5-(10-1)^5}{10^5}\right]=7.581884401$ gallons, the quantity of wine left after putting in the last gallon of wine, and, therefore, 2.418115599 gallons=quantity of water in the cask.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

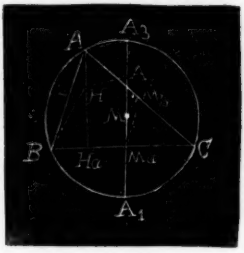
SOLUTIONS OF PROBLEMS.

71. Proposed by ROBERT J. ALEY, A. M., Ph. D., Professor of Mathematics, Indiana University, Bloomington, Indiana.

Prove by pure geometry: A perpendicular at the middle point, M_a, of the side BC of the triangle ABC meets the circumcircle in A'. On this perpendicular A'' and A''' are taken so that $M_aA''=M_aA'$ and $A''A'''=AH$. (H is the orthocenter of triangle ABC.) Prove that A''' is on the circumicircle.

Solution by F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, New Jersey, and the PROPOSER.

Let $M_aA_1=M_aA_2$, $A_2A_3=AH$, to prove A_3 on the circumference of the circle. Since A_2A_3 is a line through M, the center of the circle, the proposition is in effect to prove A_3 one extremity of the diameter through M_a.



By the conditions $AH=A_2A_3$, and is parallel to it, therefore AHA_3A_2 is a parallelogram.

Also triangles BHA and M_aMM_b are similar, hence since $2M_aM_b=AB$, we have $AH=2MM_a$.

Therefore, $A_1A_3=A_2A_3+A_2M_a+M_aA_1$.

$=AH+2M_aA_1$

$=2M_aM+2M_aA_1$

$=2(MA_1)=2r$, hence A_3 is extremity of diameter.

Q. E. D.

Also solved by *CHAS. C. CROSS*, and *J. W. SCROGGS*.
Mr. Cross furnished two different solutions.

72. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

If a line with its extremities upon two curves move in any manner whatever, (the line may vary in length), and P a point upon the line which divides it in the ratio $m:n$ describe a curve, the area of this curve will be given by the formula—

$$A=\frac{(m^2+nm)A_1+(n^2+mn)A_2-mnA_3}{(m+n)^2}.$$

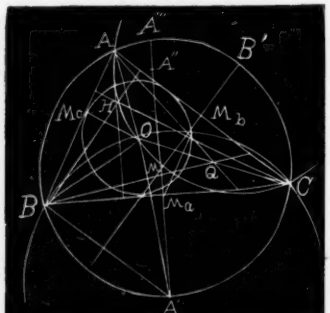
No solution of this problem has been received.

73. Proposed by ROBERT J. ALEY, A. M., Ph. D., Professor of Mathematics, Indiana University, Bloomington, Indiana.

Prove by pure geometry: (1) A', B', and C' are the middle points of the arcs BC, CA, and AB respectively. With these points as centers, circles are described passing through B and C, C and A, and A and B respectively. Prove that these circles intersect in O, the center of the incircle of the triangle ABC; (2) that O, the center of the incircle, is Nagel's point of the triangle formed by joining the middle points of the sides.

Solution by CHARLES C. CROSS, Laytonsville, Maryland, and the PROPOSER.

(1) AO cuts the circumcircle at A', for AO bisects angle A and also its subtending arc. $\measuredangle OBA'=\frac{1}{2}(A+B)$.



$\measuredangle BOA'=\frac{1}{2}(A+B)$ for it is exterior angle to triangle BOA.

$\therefore$ triangle $A'BO$ is isosceles.

$A'B=A'O$. By similar reasoning it is proved that $B'A=B'O$ and $C'A=C'O$.

$\therefore$ The circles intersect in O.

(2) It is a well known property of Nagel's point that AQ and OM_a, BQ and OM_b, CQ and OM_c are respectively parallel.

The triangle $M_aM_bM_c$ is similar to the triangle ABC.

$$\measuredangle OM_aM_c=\measuredangle QAC.$$
$$\measuredangle OM_bM_c=\measuredangle QBC.$$
$$\measuredangle OM_cM_a=\measuredangle QCA.$$

$\therefore$ O with respect to the triangle $M_aM_bM_c$, is located precisely as Q is with respect to the triangle ABC.

Hence O is Nagel's point of triangle $M_aM_bM_c$.

Also solved by *F. M. McGAW* and *G. B. M. ZERR*.

74. Proposed by ROBERT J. ALEY, A. M., Ph. D., Professor of Mathematics, Indiana University, Bloomington, Indiana.

Let O be the center of the inscribed circle. AO produced meets the circumcircle in A'. Find the ratio of AO to OA'.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics, Ohio University, Athens, Ohio.

The coördinates of A are $\left(\frac{2\triangle}{a},\ 0,\ 0\right)$; of O, $(r,\ r,\ r)$; and of A', those of the intersection of $\beta\gamma-\gamma=0$.....(1), with $a\beta\gamma+b\alpha\gamma+c\alpha\beta=0$.....(2), having the constant relation $a\alpha+b\beta+c\gamma=2\triangle$.....(3). These give for the coördinates of A' $\left(-\frac{(b+c)^2}{a^2}-\frac{2\triangle}{a},\quad \frac{(b+c)^3}{a^3}+\frac{2\triangle(b+c)}{a^2},\quad \frac{(b+c)^3}{a^3}+\frac{2\triangle(b+c)}{a^2}\right)$.

The distance d between $(\alpha_1,\ \beta_1,\ \gamma_1)$ and $(\alpha_2,\ \beta_2,\ \gamma_2)$ is given by

$$d^2=-\frac{abc}{4\triangle^2}\{a(\beta_1-\beta_2)(\gamma_1-\gamma_2)+b(\gamma_1-\gamma_2)(\alpha_1-\alpha_2)+c(\alpha_1-\alpha_2)(\beta_1-\beta_2)\}\quad ...,(4).$$

Putting $\alpha_1=(2\triangle/a)$, $\beta_1=0$, $\gamma_1=0$; $\alpha_2=\beta_2=\gamma_2=r$,

$$\overline{AO}^2=bcr(b+c-a)/2\triangle\quad .....(5).$$

Putting α_1, β_1, γ_1 equal respectively to the coördinates of A', and $\alpha_2=\beta_2=\gamma_2=r$ as before, in (4), we get an expression for $\overline{OA'}^2$.

We can then express the ratio of OA to OA'.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

The point A' is evidently the middle point of arc BC. Since $\angle A'OC=\frac{1}{2}(A+C)$ and $\angle A'CO=\frac{1}{2}(A+B)$, $OA'=A'C=A'B$.

From Ptolemy's theorem, $ACA'B$ being a cyclic quadrilateral,

$$AB\times A'C+AC\times A'B=AA'\times BC,\text{ or}$$
$$c\times OA'+b\times OA'=(AO+OA')a.$$
$$\therefore\ OA:OA'=b+c-a:a=3-a:2a.$$

Also solved by *G. B. M. ZERR* and *CHAS. C. CROSS*.

75. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

A plane passes through $(0, 0, c)$ and touches the circle $x^2 + y^2 = a^2$, $z=0$; determine the locus of the ultimate intersections of the plane.

I. Solution by the PROPOSER.

Let the plane be $Ax+By+Cz+p=0$ .....(1).

Passing through (0, 0, c), (1) gives $p=-cC$ .....(2),

and (1) becomes $Ax+By+Cz-cC=0$.....(3).

The x, y, z of (3) are those of $x^2+y^2=a^2$.....(4), $z=0$..... ...(5),

and also of $Ax+By-cC=0$.....(6).

Making (4) homogeneous by aid of (6),

$$\left[\frac{1}{a^2}-\frac{A^2}{c^2C^2}\right]\frac{x^2}{y^2}-\frac{2AB}{c^2C^2}x/y+\left[\frac{1}{a^2}-\frac{B^2}{c^2C^2}\right]=0 \quad \ldots\ldots(7).$$

For (3) to touch (7), the values of x/y from (7) must be equal, or

$$\left[\frac{1}{a^2}-\frac{A^2}{c^2C^2}\right]\left[\frac{1}{a^2}-\frac{B^2}{c^2C^2}\right]=\frac{A^2B^2}{c^4C^4} \quad \ldots\ldots(8),$$

or, $A^2/C^2+B^2/C^2-c^2/a^2=0$.....(9).

From (3), $A/C=(z-c)/x-(y/x)(B/C)$.....(10).

Substituting (10) in (9), etc.,

$$\frac{x^2+y^2}{x^2}B^2/C^2-\frac{2y(z-c)}{x^2}B/C+\frac{(z-c)^2}{x^2}-c^2/a^2=0 \quad \ldots\ldots(11),$$

a quadratic in the undetermined constant B/C, giving the envelope

$$\frac{x^2+y^2}{a^2}=\frac{(z-c)^2}{c^2} \quad \ldots\ldots(12).$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let the plane touch the circle at the point (x', y').

$\therefore \frac{xx'}{a^2}+\frac{yy'}{b^2}+z/c=1$, is the equation to the plane, but

$x'^2+y'^2=a^2$.....(1). $\therefore dy'/dx'=-x/y=-x'/y'$.....(2).

(2) in the equation to the plane gives,

$$x'=\frac{a^2x(c-z)}{c(x^2+y^2)},\quad y'=\frac{a^2y(c-z)}{c(x^2+y^2)}.$$

These values of x', y' in (1), $x^2+y^2-(a^2/c^2)(c-z)^2=0$, a cone of revolution as the locus.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

49. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

A rectangular stick of timber of known dimensions is placed upon a platform of given height in a vertical position with the center above the edge of platform, and slightly displaced from the vertical. Where and in what manner will it strike the ground?

I. Solution by the PROPOSER.

Any body rotating about the center of an end has the energy,

$$\tfrac{1}{3}(\omega^2)m(a^2+b^2).$$

If the body has fallen through the angle θ, the energy is

$$\tfrac{1}{2}b(1-\cos\theta)m.$$

$$\therefore\ 2\omega^2(a^2+b^2)=3b(1-\cos\theta)+ \quad \ldots\ldots(1).$$

The body will leave the platform when the statical pressure=centrifugal force. The pressure=$m\cos\theta$. Centrifugal force=$\tfrac{1}{2}m\omega^2b$.

$$\therefore\ \omega^2b=2\cos\theta \quad \ldots\ldots(2).$$

From (1) and (2), $\omega=\sqrt{\dfrac{6b}{4a^2+7b^2}}$ .... (3), and $\cos\theta=\dfrac{3b^2}{4a^2+7b^2}$ .....(4).

Take the edge of platform as origin. Let the axis of x be horizontal and the axis of y vertical. Resolve the angular velocity of the center of gravity into its vertical and horizontal components at the instant of the stick leaving the platform.

$$\left.\begin{aligned} V_x&=\tfrac{1}{2}b\omega\cos\theta \\ V_y&=\tfrac{1}{2}b\omega\sin\theta \end{aligned}\right\} \quad \ldots\ldots(5).$$

For the accelerations we have $\frac{d^2x}{dt^2}=0, \qquad \frac{d^2y}{dt^2}=-g.$

Then $\frac{dx}{dt}=c$, and $\frac{dy}{dt}=-gt+c_2$.

Let us begin to reckon time from the instant that the body leaves the platform.

Then $\frac{dx}{dt}=V_x$, and $\frac{dy}{dt}=-gt+V_y$.

$x=V_xt+c_3$. $y=-\frac{1}{2}gt^2+V_yt+c_4$.

When $t=0$, $x=\frac{1}{2}b\sin\theta$, and $y=\frac{1}{2}b\cos\theta$.

$$\left.\begin{aligned}&\text{Then } x=V_xt+\tfrac{1}{2}b\sin\theta\\&\text{and } y=-\tfrac{1}{2}gt^2+V_yt+\tfrac{1}{2}b\cos\theta\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

These give the motion of the center of gravity.

Call T the time taken for one end to reach the ground. Then after leaving the platform it will have rotated through the angle $T\omega$.

It therefore makes an angle $\theta+T\omega$ with the vertical.

The center of gravity has fallen,

$$Y_1=-\tfrac{1}{2}gT^2+V_yT+\tfrac{1}{2}b\cos\theta. \quad \text{Also } X_1=V_xT+\tfrac{1}{2}b\sin\theta.$$

The center of gravity will be the distance $\frac{1}{2}b\cos(\theta+T\omega)$ from the ground.

Now $Y_1+\frac{1}{2}b\cos(\theta+T\omega)=H$, the height of tower.

Or, $\frac{1}{2}b\cos(\theta+T\omega)-\frac{1}{2}gT^2+V_yT+\frac{1}{2}b\cos\theta=H$ ..... (7).

From this equation T may be determined. The horizontal distance from the foot of the tower will then be given by the equation,

$$X=X_1+\tfrac{1}{2}b\cos(\theta+T\omega)=V_xT+\tfrac{1}{2}b\sin\theta+\tfrac{1}{2}b\sin(\theta+T\omega).$$

II. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O. University, Mississippi.

The stick turns about its lower extremity until it reaches a horizontal position with an angular velocity given by $\omega^2=(3g/2a)$, $2a$ being its length.

Subsequently there are two motions which may be considered independently. One is that of rotation about the center of gravity with a constant angular velocity, ω; the other, that of translation, the center of gravity falling vertically with an initial velocity, $a\omega$.

Estimating the motion from the horizontal position, the stick is vertical when it has turned through an odd number of right angles; that is, at the end of $n\pi/2\omega$ seconds, n being any odd number. If S_v denote the distance from the level of the platform to the lowest point of the stick at the instants of verticality, the motion of translation gives,

$$S_v - a = a\omega \frac{n\pi}{2\omega} + \tfrac{1}{2}g\left(\frac{n\pi}{2\omega}\right)^2,$$

or, substituting the value of ω,

$$S_v = \{1 + \tfrac{1}{2}(\pi n)[1 + \tfrac{1}{8}(\pi n)]\}a, \; n \text{ being odd.}$$

Similarly the positions of horizontality are given by

$$S_h = \tfrac{1}{2}(\pi n)[1 + \tfrac{1}{8}(\pi n)]a, \; n \text{ being even.}$$

If any value of S_v or S_h equals D, the distance from the platform to the ground, the stick will strike the ground, in the one case vertically, in the other, horizontally.

The discussion might be continued in general terms. Instead of this, however, let $a=1$ foot, and $D=10$ feet.

Giving to n the values of 1, 2, and 3 in the proper equations, the first and second values of S_v are found to about 3.4 and 13.1, and the first value of S_h about 6.4. Consequently the stick will strike the ground in passing from a horizontal toward a vertical position.

Since in falling 6.4 feet a half revolution has been made, the time for this motion is π/ω seconds, and the velocity of the center of gravity when the stick is horizontal for the last time is $\omega + g(\pi/\omega)$, remembering that $a=1$. If the stick turns through an angle θ before striking the ground, the center of gravity falls through $(3.6 - \sin\theta)$ feet in θ/ω seconds, giving the equation,

$$3.6 - \sin\theta = [\omega + g(\pi/\omega)](\theta/\omega) + \tfrac{1}{2}g[(\theta/\omega)]^2,$$

which reduces to $3.6 - \sin\theta = 3.1\theta + \frac{1}{3}\theta^2$, approximately ; from which $\theta = 48^\circ\ 20'$, about.

The horizontal distance from the edge of the platform to the point at which the stick touches the ground is $1 + \cos\theta$, or 1 foot, 8 inches, approximately.

θ is, of course, the inclination of the stick to the horizontal at the instant of contact with the ground.

In the last part of this work the thickness of the stick has been neglected.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

49. Proposed by EDMUND FISH, Hillsboro, Illinois.

A rectangular field, whose length and breadth in rods are in whole numbers, is enclosed with a fence and subdivided by fences on both diagonals, the total length of the fences being 2204 rods; required the sides and area.

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.; O. S. WESTCOTT, North Division High School, Chicago, and the PROPOSER.

Let $2xy$, x^2-y^2, and x^2+y^2 be the length, breadth and diagonal of the field, respectively; then $2x^2+2xy=1102$.

$\therefore x^2+xy=551$; whence $y=\frac{551}{x}-x$, $=\frac{19\times 29}{x}-x$.

As x and y are known to be integral, $551/x$ must be integral, which can only be when $x=19$. Hence $y=10$.

$\therefore 2xy=380$; and $x^2-y^2=261$, breadth.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let x=length, y=breadth; then $2x+2y+2\sqrt{(x^2+y^2)}=2204$.

$\therefore 607202+xy=1102(x+y)$. $\therefore 1102-y=1102(551-y)/x$.

Let $1102-y=z$, then $z=1102(x-551)/x$.

$\therefore x=1102-(2.19^2.29^2/z)$. $\therefore z=29^2$. $\therefore x=380$, $y=261$.

Area=99180 square rods, =619 acres, 14 square rods.

III. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; JOSIAH H. DRUMMOND, LL. D., Portland, Maine; A. H. HOLMES, Brunswick, Maine; and P. S. BERG, Larimore, North Dakota.

As the field is a rectangle, the diagonals are equal, and the fences form the sides of two equal right triangles of which the legs and hypotenuse are respectively the sides and diagonal of the field.

Let a and b be the sides and c the diagonal of the field. Then $2a+2b+2c=2204$, and $a^2+b^2=c^2$. From the identity $(2mn)^2+(m^2-n^2)^2=(m^2+n^2)^2$, the formula for finding integral sides of right triangles, take $a=2mn$, $b=m^2-n^2$, and $c=m^2+n^2$. Then $2a+2b+2c=4m(m+n)=2204$. Whence $m(m+n)=551$. We now separate 551 into two factors, making the larger factor equal $m+n$, and the smaller equal m.

$551=19\times 29$. Then $m+n=29$ and $m=19$; whence $n=10$. Substituting these values of m and n in the values for a, b, and c, we obtain $a=2mn=380=$ the length of the field; $b=m^2-n^2=261=$the breadth of the field; and $c=m^2+n^2=461=$the diagonal of the field. The area$=380\times 261=99180$ square rods$=619\frac{7}{8}$ acres.

Also solved by *A. H. BELL*.

50. Proposed by SYLVESTER ROBINS, North Branch Depot, New Jersey.

The edges of a rectangular parallelopiped are within 1 of the proportion 2 : 3 : 9, and they are $2x\pm1$, $3x$ and $9x$, $(2x\mp1)^2+(3x)^2+(9x)^2$=the diagonal squared$=94x^2\mp4x+1=\square$. To find four integral values for x.

I. Solution by A. H. HOLMES, Box 963, Brunswick, Maine.

We may put it in the form : $90x^2+(2x\pm1)^2=\square$, or

$$m^2x^2-(m^2-90)x^2+(2x\pm1)^2=\square.$$

$\therefore 2m(2x\pm1)=(m^2-90)x$; $4mx\pm2m=m^2x-90x$.

$\therefore x=\pm(2m/(m^2-4m-90)$.

Let $m=nx$. Then $n^2x^2-4nx-90=\pm2n$; $n^2x^2-4nx+4=94\pm2n$.

Take plus sign and let $n=3$. $\therefore 3x=2+10=12$. $\therefore x=4$.

Now let $n=a/b^2$. $a^2x^2/b^4-4ax/b^2+4=94\pm2a/b^2=(94b^2\pm2a)/b^2$.

Now take $b=3$. $\therefore a=5/2$ and $a/b^2=5/18$.

$\therefore 5x/18=2+29/3$. $5x=36+174=210$. $\therefore x=42$.

Now let $b=10$. $\therefore a=9/2$ and $a/b^2=9/200$.

$\therefore 9x/200=2+97/10$. $9x=400+1940=2340$. $\therefore x=260$.

Now let $b=23$. $\therefore a=-3/2$ and $a/b^2=-3/1058$.

$\therefore -3x/1058=2-223/23$, or $3x=8142$. $\therefore x=2714$.

For $x=$ 4 we have : $94x^2+4x+1=\square$.

For $x=$ 42 we have : $94x^2-4x+1=\square$.

For $x=$ 260 we have : $94x^2+4x+1=\square$.

For $x=2714$ we have : $94x^2-4x+1=\square$.

II. Solution by A. H. BELL, Hillsboro, Illinois.

The equation readily reduces to : $t^2-94y^2=-90$.....(1), and $x=(t\mp2)/94$ ....(2). (1)$\div9$ gives $t'^2-94y'^2=-10$, and $t=3t'$, $y=3y'$.....(3).

One cycle.

1/94. No. of Frac's:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16:
Quotients complete Denom'rs	1 : 13	6	5	9	10	3	15	2	15	3	10	9	5	6	13	1:
Quotients	9 : 1,	2,	3,	1,	1,	5,	1,	(8),	1,	5,	1,	1	3	2	1	18:
Convergents $\frac{b}{a}$, $\frac{y}{t}$:	$\frac{10}{1}$,	$\frac{29}{3}$,	$\frac{97}{10}$,	$\frac{126}{13}$,	$\frac{223}{23}$,	$\frac{1241}{128}$,	$\frac{1464}{151}$,	$\frac{12557}{1295}$,	$\frac{14417}{1487}$,	$\frac{85038}{8771}$,	$\frac{99455}{10258}$,	$\frac{184429}{19029}$,	$\frac{652934}{67345}$,	$\frac{1490361}{153719}$,	$\frac{2143295}{221064}$.—	

The convergents preceding the denominators, 10 of the complete quotients $=t'/y'=126/13$ and $85038/8771$ .....(4), as they are even fractions.

$\therefore$ answer the -10 of (3). To obtain other values of t' and y', take $v^2-94u^2=1$.....(5).

$$\left.\begin{array}{l}(3)\times(5) \text{ and } \pm188t'uvy' \ . \ (t'v\pm94uy')^2-94(t'u\pm vy')^2=-10 \\ \text{or, } t_n'^2-94y_n'^2=-10\end{array}\right\} \ldots\ldots\ldots\ldots (6).$$

The smallest integral values for $v/u=2143295/221064$, but as fractional

values for t' and y' can be used as shown in (3), to obtain these we solve (5).

Let $v=v'/z$ and $u=u'/z$; then $v'^2-z^2=94u'^2$ .....(7).

Now let $u^2=pq$ and let 94=any two factors, then (7) can be made

$$\left.\begin{matrix} v'+z=p^2 \text{ or } 2p^2 \\ v'-z=94q^2 \text{ or } 47q^2 \end{matrix}\right\}$$

add and subtract, etc. $v'=p^2+94q^2$ or $2p^2+47q^2$; $z=p^2-97q^2$ or $2p^2-47q^2$; $u'=2pq$.....(8).

In the right-hand values if $p=5$ and $q=1$, $v'=97$; $z=3$; $u'=10$. There are an infinite number of values but these are the only ones admissible.

(7) $v=97/3$ and $u=10/3$; substituting these along with those of (4) separately in (6) we have $t_n'=2/3$ and 24442/3 ; and $t_n'=3946/3$ and 16493426/3 with those in (4), will make six values for t', and now in (3) and (2) $x=0$, 4, -42, 260, -2714, and 175462, etc. The sign=side$(2x\pm1)$. $y=94$, 39, 407, 2521, 26313.

III. Solution by the PROPOSER.

This problem is suggested by a remark in No. 5, Vol. I.: "$x^2-94y^2=\pm1$; this is the most difficult number under 100."

1. Find initial terms in that infinite series of rational rectangular solids where the edges of each term are in proportion as 2 : 3 : 9, within 1 in the *thickness*.

Let $2x\pm1$, $3x$ and $9x$ be the edges ; then $94x^2\pm4x+1=\square=(mx\pm1)^2=m^2x^2\pm2mx+1$. $x=(\pm2m\mp4)/(94-m^2)$.

Say $m=\sqrt{(94)}=9/1$, 10/1, 29/3, 97/10, 126/13, 223/23, 1241/128, 1464/151, etc.

When $m=$ / Then $x=$	10 / 4	29/3 / 42	97/10 / 260	223/2x / 2714	
$2x\pm1=$	9	83	521	5427	Thickness.
$3x=$	12	126	780	8142	Width.
$9x=$	36	378	2340	24426	Length.
$\sqrt{(94x^2\pm4x+1)}=$	39	407	2521	26313	Solid diagonal.

2. Find first term in an infinite series of rational parallelopipeds where the dimensions of every solid are in proportion as 2 : 3 : 9, within 1 in the *width*.

Let $2x$, $3x\pm1$ and $9x$ represent the edges. Then $94x^2\pm6x+1=\square=(mx\pm1)^2=m^2x^2\pm2mx+1$. Whence $x=(2m\mp6)/(94-m^2)$, $m=\sqrt{(94)}=9$, 10, 29/3, 97/10, 126/13, etc.

$m=$	29/3	126/13
$x=$	24	429
$2x=$	48	858
$3x\pm1=$	73	1286
$9x=$	216	3861
Solid diagonal=	233	4159

3. Find a term in an infinite series of rational parallelopipeds where the edges are in proportion as 2 : 3 : 9, within unity in *length*.

Let $2x$, $3x$, and $9x\pm1$ be the edges. $94x^2\pm18x+1=\square=(mx\pm1)^2=m^2x^2\pm2mx+1$. $x=(2m\mp18)/(94-m^2)$. Substitute $m=1464/151$, and $x=15855$, $2x=31710$, $3x=47565$, $9x-1=142694$.

Proof: $31710^2+47565^2+142694^2=153719^2$.

4. Find some term in an infinite series of rational parallelopipeds where the dimensions come within 1 unit in the *thickness* of being in proportion as 3 : 6 : 7.

Let edges be $3x\pm1$, $6x$ and $7x$. $94x^2\pm6x+1=\square=(mx\pm1)^2=m^2x^2\pm2mx+1$. $x=(2m\mp6)/(94-m^2)$.

When $m=29/3$	$m=126/33$	
$x=$ 24	$x=$ 429	
$3x\pm1=$ 144	$3x\pm1=$ 1286	etc.
$6x=$ 144	$6x=$ 2574	
$7x=$ 168	$7x=$ 3003	
S. d. = 233	S. d. = 4159	

Proof: $73^2+144^2+168^2=233^2$.

5. Find some term in an infinite series of rational rectangular solids where the edges come within 1 unit in the *width* of being in the proportion of 3 : 6 : 7. Let the edges be represented by $3x$, $6x\pm1$ and $7x$. Then $94x^2\pm12x+1=\square=(mx\pm1)^2=m^2x^2+2mx+1$. $x=(2m\mp12)/(94-m^2)$. When $m=\sqrt{94}$ .....1464/151. Then $x=84258$ or 357870.

$3x=252774$	or $3x=1073610$
$6x-1=505547$	$6x+1=2147221$
$7x=589806$	$7x=2505090$
Diagonal = 816911	Diagonal = 3469679

6. Find a term in that infinite series of rational parallelopipeds wherein the edges of every solid are within unity in the *length* of being in proportion to each other as 3 : 6 : 7.

$(3x)^2+(6x)^2+(7x\pm1)^2=94x^2\pm14x+1=\square=(mx\pm1)^2$.

$94x\pm14=m^2x\pm2m$. $x=(2m\mp14)/(94-m^2)$. $m=\sqrt{94}$. Now when $m=29/3$, $x=60$, $3x=180$, $6x=360$, $7x-1=419$.

$180^2+360^2+419^2=581^2$.

Also solved by *J. H. DRUMMOND.*

51. Proposed by H. C. WILKES, Skull Run, West Virginia.

The difference between the roots of two successive triangular square numbers, [i. e. triangular numbers that are also square numbers], equals the sum of two successive integral numbers, the sum of whose squares will be a square number. Demonstrate. Or, if s and t be the roots of any two successive triangular number that are also square numbers, prove that $t-s=2n+1$, where $n^2(n+1)^2=\square$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

$\frac{n(n+1)}{2}$ is a square when $n=\frac{(1+\sqrt{2})^{2m}+(1-\sqrt{2})^{2m}-2}{4}$.

$$\therefore \pm\sqrt{\frac{n(n+1)}{2}} = \pm\left\{\frac{(1+\sqrt{2})^{2m}-(1-\sqrt{2})^{2m}}{4\sqrt{2}}\right\} \quad \ldots\ldots\ldots\ldots (1).$$

$$\pm\sqrt{\frac{n'(n'+1)}{2}} = \pm\left\{\frac{(1+\sqrt{2})^{2m+2}-(1-\sqrt{2})^{2m+2}}{4\sqrt{2}}\right\} \quad \ldots\ldots\ldots\ldots (2).$$

Taking (2)+ and (1)−, and then taking their difference, we easily get,

$$\frac{(1+\sqrt{2})^{2m+2}-(1-\sqrt{2})^{2m+2}}{4\sqrt{2}} + \frac{(1+\sqrt{2})^{2m}-(1-\sqrt{2})^{2m}}{4\sqrt{2}} = 2y+1.$$

$$\therefore \frac{(1+\sqrt{2})^{2m+1}+(1-\sqrt{2})^{2m+1}}{2} = 2y+1.$$

$$\therefore \left\{\frac{(1+\sqrt{2})^{2m+1}+(1-\sqrt{2})^{2m+1}}{4} - \tfrac{1}{2}\right\}^2 +$$

$$\left\{\frac{(1+\sqrt{2})^{2m+1}+(1-\sqrt{2})^{2m+1}}{4} + \tfrac{1}{2}\right\}^2 = y^2+(y+1)^2.$$

$$\therefore 2\left\{\frac{(1+\sqrt{2})^{2m+1}+(1-\sqrt{2})^{2m+1}}{4}\right\}^2 + \tfrac{1}{2} = y^2+(y+1)^2.$$

$$\therefore \left\{\frac{(1+\sqrt{2})^{2m+1}-(1-\sqrt{2})^{2m+1}}{2\sqrt{2}}\right\}^2 = y^2+(1y+1)^2.$$

In above m can have any positive integral value.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

This problem is true if we read "The sum of" instead of "The difference between." It might also be stated as follows: The difference between the roots of two successive triangular square numbers equals a number whose square is the sum of the squares of two successive integral numbers.

From Solution III of Problem 36, Vol. III., No. 3, page 82, we find that when one of the triangular square numbers is $n(n+1)/2$, the next in order, in terms of n, is $\left(2n+1+3\sqrt{\frac{n(n+1)}{2}}\right)^2$.

The *difference* of the two roots is $2n+1+2\sqrt{\frac{n(n+1)}{2}}$.

The *sum* of the two roots is $2n+1+4\sqrt{\frac{n(n+1)}{2}}$, which equals the sum of the two consecutive integral numbers, $n+2\sqrt{\frac{n(n+1)}{2}}$ and $n+1+2\sqrt{\frac{n(n+1)}{2}}$.

But $\left(n+2\sqrt{\frac{n(n+1)}{2}}\right)^2 + \left(n+1+2\sqrt{\frac{n(n+1)}{2}}\right)^2 = 6n^2+6n+1+(8n+4)\sqrt{\frac{n(n+1)}{2}}$

which equals the square of the *difference* of the two roots, or

$$\left(2n+1+2\sqrt{\frac{n(n+1)}{2}}\right)^2.$$

Illustration.—From the series of triangular square numbers, 1^2, 6^2, 35^2, 204^2, 1189^2, etc., take 6 and 35. $35-6=29$; $35+6=41=20+21$; $20^2+21^2=29^2$.

This problem and problems No. 45, (Vol. III., No. 5, page 153), and No. 36, of Diophantine Analysis, are very closely related.

Also solved by the *PROPOSER*.

52. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbiam University, Washington, D. C.

Prove that a "magic square" of nine integral elements, whose rows, columns, and diagonals have a constant sum, is only possible when this sum is a multiple of three.

I. Solution by M. W. HASKELL, M. A., Ph. D., Associate Professor of Mathematics, University of California, Berkeley, California.

Let the magic square be $\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline g & h & k \\ \hline \end{array}$ and let S be the constant sum.

Then $S=a+b+c=d+e+f=g+h+k=a+d+g=b+e+h=c+f+k=a+e+k=c+e+g$.

Adding these all together, we have $8S=3a+2b+3c+2d+4e+2f+3g+2h+3k=3(a+c+g+k)+2(b+e+h)+2(d+e+f)$. But the last two quantities in parenthesis are each$=S$. Hence $4S=3(a+c+g+k)$, and S is a multiple of 3.

II. Solution by —— (Paper Unsigned.)

Suppose the numbers occupying the magic square to be a, b, c, d, e, f, g, h, k. Now $a+e+k=b+e+h=c+e+g=S$.

$\therefore a+k\equiv k \pmod 3$, $b+h\equiv k \pmod 3$, $c+g\equiv k \pmod 3$, where $S-e\equiv k \pmod 3$.

Adding the congruences, $(a+b+c)+(g+h+k)\equiv 0 \pmod 3$. Or, since $(a+b+c)+(g+h+k)\equiv 0 \pmod 3$, $2S\equiv 0 \pmod 3$.

Multiply by 2, and divide by 3, and the result is $S\equiv 0$. Q. E. D.

III. Solution by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana.

Let the rows of the "square" be a, b, c; x, y, z; and l, m, n, and let the constant sum be k. We have to show that $k/3$ is integral. We have $a+y+n=k$; $b+y+m=k$; $l+y+c=k$. Add, and we have $(a+b+c)+(l+m+n)+3y=3k$, that is, $2k+3y=3k$.

$\therefore 3y=k$. $\therefore y=k/3$. But y is integral. $\therefore k/3$ is integral.

Also solved by *M. A. GRUBER* and *G. B. M. ZERR*.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

48. Proposed by P. H. PHILBRICK, C. E., Pineville, Louisiana.

A, B, C, D, and E play with dice, each throwing three, three successive times, for a stake a. A, B, and C throw; C throwing the highest, 52. What is his expectation?

I. Solution by the PROPOSER.

If D or E or both throw 52, C gets but a part of the stake. If D or E or both throw 53 or 54, C gets none of the stake.

$52=18+18+16=18+17+17$. $53=18+18+17$. $54=18+18+18$.

The chance of throwing 16 at a single throw is $\frac{6}{216}$.

The chance of throwing 17 at a single throw is $\frac{3}{216}$.

The chance of throwing 18 at a single throw is $\frac{1}{216}$.

Hence since D may throw 16 (or 18) at any one of the three throws, his chance of throwing 52 at three throws is $3(\frac{1}{216}\times\frac{1}{216}\times\frac{6}{216})+3(\frac{1}{216}\times\frac{1}{216}\times\frac{3}{216})=p_1$ say. E has the same chance of reaching the same result. The chance that D (or E) will not throw 52 is $(1-p_1)$; and the chance that D or E will throw 52 and the others not is $p_1(1-p_1)$, in which case the expectation is $p_1(1-p_1)\frac{1}{2}a$.

The chance that D, and E also, will throw 52 is p_1^2, in which case their joint expectation is $p_1^2\frac{2}{3}a$. Hence the expectation of D or E or of both, coming from throwing 52 is, $2p_1(1-p_1)\frac{1}{2}a+p_1^2\frac{2}{3}a=p_1(3-p_1)\frac{1}{3}a$.

The chance of D or E throwing 53 is, $3(\frac{1}{216}\times\frac{1}{216}\times\frac{3}{216})=p_2$; and the chance that one or both will throw 53 is, $2p_2(1-p_2)+p_2^2=p_2(2-p_2$; and their joint expectation is, $p_2(2-p_2)a$.

The chance that D or E will throw 54 is $(\frac{1}{216}+\frac{1}{216}+\frac{1}{216})=p_3$); and the chance that one or both will throw 54 is, $2p_3(1-p_3)+p_3^2=p_3(2-p_3)$; and their joint expectation is, $p_3(2-p_3)a$. Hence C's expectation is,

$$\{1-\tfrac{1}{3}[p_1(3-p_1)]-p_2(2-p_2)-p_3(2-p_3)\}a=(1+325p_3^2-47p_3)a.$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, and J. SCHEFFER, A. M., Hagerstown, Maryland.

D and E may each throw 52, 53, or 54.

52 can be thrown as follows: (6, 6, 6), (6, 6, 5), (6, 6, 5); (6, 6, 6), (6, 6, 6), (6, 6, 4).

53 can be thrown as follows: (6, 6, 6), (6, 6, 6), (6, 6, 5).

54 can be thrown as follows: (6, 6, 6), (6, 6, 6), (6, 6, 6).

D's chance of throwing 52, 53, or 54 is,

$$p=\frac{9}{(216)^3}+\frac{3}{216^3}+\frac{3}{216^3}+\frac{1}{216^3}=\frac{16}{(216)^3}=\frac{2^4}{6^9}.$$

$1-p=1-2^4/6^9=\frac{1}{1}\frac{0}{0}\frac{0}{0}\frac{7}{7}\frac{7}{1}\frac{6}{6}\frac{8}{9}\frac{0}{6}=P$, chance that D will throw less than 52.

P^2=chance that D and E will both throw less.

$\therefore aP^2=C$'s expectation on the supposition that C wins and no ties.

49. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

A square whose side is $2a$ and an equilateral triangle whose altitude is $3a$ are fastened together at their centers, but otherwise free to move. If they are thrown on a floor at random, what is the average area common to both ?

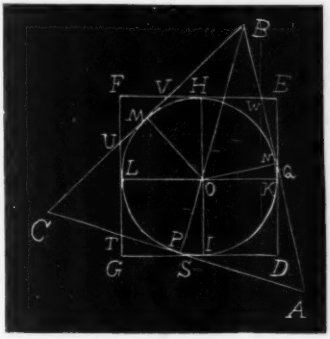
Solution by HENRY HEATON, M. Sc., Atlantic, Iowa, and the PROPOSER.

In the figure let O be the common center of the square and the triangle.

Then $OK=ON=OH=OM=OL=OP=OI=a$.

Let the triangle $KON=2\theta$.

Then $\angle NOH=\frac{1}{2}\pi-2\theta$, $\angle HOM=\frac{2}{3}\pi-(\frac{1}{2}\pi-2\theta)=\frac{1}{6}\pi+2\theta$, $\angle MOL=\frac{1}{2}\pi-(\frac{1}{6}\pi+2\theta)=\frac{1}{3}\pi-2\theta$, $\angle LOP=\frac{2}{3}\pi-(\frac{1}{3}\pi-2\theta)=\frac{1}{3}\pi+2\theta$, and $\angle POI=\frac{1}{2}\pi-(\frac{1}{3}\pi+2\theta)=\frac{1}{6}\pi-2\theta$.



Area of surface, $KONQ$, $=a^2\tan\theta$;

area of surface, $NOHW$, $=a^2\tan(\frac{1}{4}\pi-\theta)$;

area of surface, $HOMV$, $=a^2\tan(1\frac{1}{2}\pi+\theta)$;

area of surface, $MOLU$, $=a^2\tan(\frac{1}{6}\pi-\theta)$;

area of surface, $KOPT$, $=a^2\tan(\frac{1}{6}\pi+\theta)$;

area of surface, $POIS$, $=a^2\tan(1\frac{1}{2}\pi-\theta)$;

area of square, $OKDI$, $=a^2$.

Hence the area common to the square and triangle is

$$S=a^2[1+\tan\theta+\tan(\tfrac{1}{4}\pi-\theta)+\tan(1\tfrac{1}{2}\pi+\theta)+\tan(\tfrac{1}{6}\pi-\theta)+\tan(\tfrac{1}{6}\pi+\theta)+\tan(1\tfrac{1}{2}\pi-\theta)].$$

The positions for $\theta>1\frac{1}{2}\pi$ are exact repetitions of those for $\theta<1\frac{1}{2}\pi$.

Hence the required average area is

$$J=\int_0^{1\frac{1}{2}\pi} Sd\theta\div\int_0^{1\frac{1}{2}\pi} d\theta=\frac{12a^2}{\pi}\int_0^{1\frac{1}{2}\pi}\Big[1+\tan\theta+\tan(\tfrac{1}{4}\pi-\theta)+\tan(1\tfrac{1}{2}\pi+\theta)+$$

$$\tan(\tfrac{1}{6}\pi-\theta)+\tan(\tfrac{1}{6}\pi+\theta)+\tan(1\tfrac{1}{2}\pi-\theta)\Big]d\theta=a^2\Big[1+\frac{12}{\pi}\log_e 2\Big].$$

This problem was also solved in a very excellent manner by G. B. M. Zerr.

50. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Find (1), the average length of all straight lines having a given direction, between 0 and a; (2), the average length of chords drawn from one extremity of the diameter a of a semi-circle to all points in the semi-circumference; and (3), find the average area of all triangles formed by a straight line of constant length a sliding between two straight lines at right angles.

Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

(1). Let x=length of one of the straight lines. Then the required average is $\int_0^a x\,dx \div \int_0^a dx = \frac{1}{2}a$.

(2). Let θ=angle between the chord and diameter. Then the length of the chord is $2a\cos\theta$, and the average length of all chords is

$$2a\int_0^{\frac{1}{2}\pi} \cos\theta\,d\theta \div \int_0^{\frac{1}{2}\pi} d\theta = 4a/\pi.$$

(3). Let θ=the angle between the sliding line and one of the fixed ones. The area of the triangle is $(\frac{1}{2}a^2)\sin\theta\cos\theta$. The average area of all such triangles depends upon circumstances. If the areas be taken at equal angular intervals, the required average is $A_1 = \int_0^{\frac{1}{2}\pi} (\frac{1}{2}a^2)\sin\theta\cos\theta\,d\theta \div \int_0^{\frac{1}{2}\pi} d\theta = a^2/2\pi$.

If the areas be taken at equal intervals as measured on one of the fixed lines along which the end of the sliding line moves, the average area is

$$A_2 = \int_0^{\frac{1}{2}\pi} (\tfrac{1}{2}a^2)\sin\theta\cos\theta\,d(a\sin\theta) \div \int_0^{\frac{1}{2}\pi} d(a\sin\theta) = \tfrac{1}{2}a^2 \int_0^{\frac{1}{2}\pi} \sin\theta\cos\theta\,d\theta = \tfrac{1}{4}a^2.$$

This is but a repetiton of Problem 26, about which there has been so much controversy. In the absence of a distinct statement as to the intervals at which the areas shall be taken I see no reason for preferring either of the above solutions to the other.

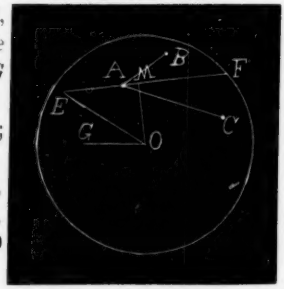
The editor quotes me as saying that there is no correct solution of the problem. I must have failed in expressing myself clearly, for my position is that in such problems no solution can be considered a full one that does not discuss all possible cases.

51. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Three points are taken at random in a sphere and a plane passed through them. Find the average volume of the segment cut off from the sphere.

Solution by the PROPOSER.

Let A, B, C be the three random points, EF the diameter of the section of the sphere made by a plane through A, B, C; M the center of this section; O the center of the sphere; OG a line such that AB is parallel to the plane MOG.



Let $OE=r$, $MA=x$, $AB=y$, $AC=z$, $\angle EOM=\theta$, $\angle BAM=\varphi$, $\angle CAM=\psi$, $\angle MOG=\lambda$, the angle the plane MOG makes with some fixed plane through OG $=\rho$.

The element of the sphere at A is $r\sin\theta d\theta . 2\pi x dx$; at B, $y^2 dy d\varphi d\lambda$; at C, $\sin(\varphi+\psi)\sin\lambda z^2 dz d\psi d\rho$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of x, 0 and $r\sin\theta$, and tripled; of φ, $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; of ψ, $-\varphi$ and $\frac{1}{2}\pi$, and doubled; of λ, 0 and π; of ρ, 0 and 2π; of y, 0 and $2x\cos\varphi$; of z, 0 and $2x\cos\psi$.

Let $r\sin\theta=x'$, $2x\cos\varphi=y'$, $2x\cos\psi=z'$, $V=$volume of segment.
$OM=r\cos\theta$. $\therefore$ The height of the segment is $r(1-\cos\theta)=2r\sin^2\frac{1}{2}\theta$.
$\therefore\ V=\frac{1}{3}\pi r^3\sin^4\frac{1}{2}\theta(3-2\sin^2\frac{1}{2}\theta)$.

Since the whole number of ways the three points can be taken is $(\frac{4}{3}\pi r^3)^3$, the required average is,

$$\Delta=\frac{6.3^3}{64\pi^3r^9}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{y'}\int_0^{z'} Vr\sin\theta d\theta 2\pi x dx \times\sin(\varphi+\psi)d\varphi d\psi\sin\lambda d\lambda d\rho y^2 dy z^2 dz.$$

$$\Delta=\frac{27}{2\pi^2r^8}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{y'} V\sin\theta\sin(\varphi+\psi)\cos^3\psi\sin\lambda \times d\theta x^4 dx d\varphi d\psi d\lambda d\rho y^2 dy$$

$$=\frac{36}{\pi^2r^8}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi} V\sin\theta\sin(\varphi+\psi)\cos^3\varphi\cos^3\psi\sin\lambda x^7 d\theta dx \times d\varphi d\psi d\lambda d\rho$$

$$=\frac{144}{\pi r^8}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi} V\sin\theta\sin(\varphi+\psi)\cos^3\varphi\cos^3\psi x^7 d\theta dx d\varphi d\psi$$

$$=\frac{18}{\pi r^8}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} V\sin\theta[3(\tfrac{1}{2}\pi+\varphi)\sin\varphi+2\cos\varphi+\sin^2\varphi\cos\varphi]\times\cos^3\varphi x^7 d\theta dx d\varphi$$

$$=\frac{315}{16\pi r^8}\int_0^{\frac{1}{2}\pi}\int_0^{x'} V\sin\theta x^7 d\theta dx$$

$$=\frac{105r^3}{32}\int_0^{\frac{1}{2}\pi}\sin^4\tfrac{1}{2}\theta(3-2\sin^2\tfrac{1}{2}\theta)\sin^9\theta d\theta=\tfrac{4}{3}r^3.$$

This is the average volume of the lesser segment.
$\frac{4}{3}r^3(\pi-1)=$average volume of greater.

EDITORIALS.

The MONTHLY will not appear during the months of July and August, but the August-September number will appear about the first of September.

We are pleased to state that our valued contributor, Dr. G. B. M. Zerr, has been called to the presidency of The Russell College, Lebanon, Va. May success, as we know it will, follow him in his new field of work.

The MONTHLY is now sorely in need of funds to carry it on further. Will those of our subscribers who are in arrears remit the amount of their subscriptions at once, so that no delay may be caused through lack of funds in getting out our next issue?

The degree of Doctor of Philosophy was conferred June 9th, by the University of Pennsylvania, on Prof. Robert J. Aley, the subject of his thesis being, "Some Contributions to the Geometry of the Triangle." We congratulate Dr. Aley on having received this degree as it is not an honorary one, but was obtained by actual work done at the University during the past year.

We are sorry that we were obliged to disappoint our readers in failing to give in the May number of the MONTHLY, the first of a series of articles on Lie's Transformation groups, by Dr. Edgar Odell Lovett. Owing to some unavoidable circumstances, Dr. Lovett was unable to prepare the articles, but he assures us that he will have his first article ready for the August-September number. We shall look forward with a good deal of interest for the appearance of the next number.

It has been proposed that the number of pages of the MONTHLY be increased from 32 to 50, half the number of which shall be devoted to papers and the other half to the solutions of problems, and the price of subscription raised to $5. per year. We shall be pleased to hear from every one of our subscribers in regard to this matter, that in case the proposition meets with the necessary endorsement it may be carried into execution at the beginning of the fifth volume. We are at all times open to advice and suggestions from our readers and no pains will be spared on our part to increase the usefulness of our journal.

The University of Chicago, Summer, 1897. The following mathematical courses will be offered :—By Professor Moore : Abstract groups ; Projective geometry.—By Professor Bolza : Hyperelliptic functions ; Advanced integral calculus.—By Dr. Lovett : The geometry of Lie's transformation-groups.—By Dr. Young : [1]Conferences on mathematical pedagogy ; [1]Determinants ; Culture Calculus ; [2]Plane trigonometry.—By Mr. Slaught : Integral Calculus ; College algebra. The courses are four or five hours weekly for twelve weeks from July 1, 1897 ; the two courses marked 1 are, however, only for the first six weeks, and

the course marked 2 is ten hours weekly for the second six weeks. Those who expect to work in mathematics at the University of Chicago during the coming summer, as well as those who desire further information, are requested to communicate with Professor Moore.

It was our intention to have appear in this issue a group of some of our contributors, but it was impossible for us to make all the necessary arrangements without delaying this number. So we have decided to have our group in the August-September number.

We are indebted to Dr. Artemas Martin for pamphlet copies of his valuable papers on "Formulas for the Sides of Rational Plane Triangles," and "A Method of Finding, without Tables, the Number Corresponding to a given Logarithm." These papers will appear in Vol. II., No. 11 of the *Mathematical Magazine.*

We have received a copy in pamphlet form of "Transcendental Numbers," by Prof. Heinrich Weber. Translated into English by Prof. W. W. Beman. Reprinted from the *Bulletin of the American Mathematical Society.* Thanks are due Professor Beman for giving us this reproduction in English of this very interesting and valuable paper.

Ginn & Co. announce for June a *Higher Arithmetic* by Wooster Woodruff Beman, of the University of Michigan, and David Eugene Smith, of the Michigan State Normal School. Teachers will await with much interest this new work on arithmetic by these well-known authors. The same publishers announce as ready "*An Elementary Arithmetic,*" by William W. Speer, being the second book of this new series.

BOOKS AND PERIODICALS.

Differential Equations. By D. A. Murray, Ph. D., of the Department of Mathematics in Cornell University. Price $1.90. 230 pages. New York and London: Longmans, Green & Co. 1897.

This work aims to meet the needs of students of physics and engineering who wish to use the subject as a tool, as well as of those students who have more time to give to the general theory and who wish to proceed to the study of the higher mathematics. For the first class, the theoretical explanations have been given as briefly as is consistent with clearness and in most cases the examples have been worked in full detail. In addition, two chapters have been introduced dealing with geometrical and physical problems. For the second class of students, notes have been inserted in the latter part of the book giving the demonstration of additional theorems and more vigorous proofs of theorems partially proved in the first part of the book. Interesting historical and biographical notes have been given in proper places, and many references are made to sources where fuller explanations and developments than the scope of the work allows may be found. We commend the book as providing an excellent introductory course in Differential Equations. J. M. C.

Analytical Geometry. By F. R. Bailey, A. M., and F. S. Woods, Ph. D., Assistant Professors of Mathematics in Massachusetts Institute of Technology. 371 pages. Boston and London: Ginn & Co. 1897.

This book is intended primarily for students in colleges and technical schools. The treatment of subjects included has been complete and rigorous. There are no important departures in method of treatment, but we notice that more space than is usual has been given to the more general form of the equations of the first and second degrees; that the equations of the conics have been derived from a single definition and then by translation of the origin equations of the second degree, wanting the xy term, are handled; and that only later the general equation of the second degree is fully discussed. In solid geometry the treatment is very satisfactory. The examples are numerous and well chosen. No use is made of determinants or calculus—a feature which many will commend and others criticise. Altogether the book is undoubtedly a good one and it should prove a useful text.

J. M. C.

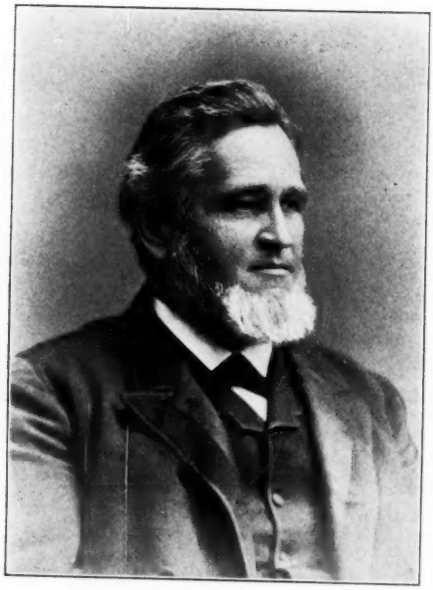
Higher Algebra. By George Lilley, Ph. D., LL. D., Ex-President South Dakota College. 504 pages. Silver, Burdett & Co., New York, Boston and Chicago. 1894.

The first 400 pages are the same as the author's "Elements." As the book only professes to cover the ground required for admission to colleges and universities, this feature is not so objectionable as it would be in a work intended for college and university use. Under the chapter on "Theory of Limits," there are several features which invite attention, such as the proof of the Theory; the sum of an infinite decreasing Geometrical Series; the invention of a symbol to represent an Infinitesimal, etc. However, to our mind the author's interpretation of the for $a/0$, or 0 *as a divisor*, is objectionable, and the proof that, in general, $a/0=0$, defective. The proof as given is,

$$\frac{12}{+2}=6,\ \frac{12}{+1}=12,\ \frac{12}{0}=0,\ \frac{12}{-1}=-12,\ \frac{12}{-2}=-6,\ \text{etc.},$$

where the quotient, 0, means that there is *no number of times zero* that the divisor, 0, can be subtracted from 12 and leave *zero*. It would misrepresent the author's position not to add that he invents a new symbol to represent an infinitesimal and shows that a (an infinitesimal)$=\infty$, and he would not confound the 0, arising from dividing a by *infinity*, with the *absolute zero*. nor *perhaps* the absolute zero with the zero, meaning "no number of times," in the quotient $a/0=0$. In interpreting the result, $t=a/0$, in Clairaut's problem of the Couriers, he would say, as there is no number of times zero that subtracted from a leaves *zero*, so there is no number of hours when they have been or will be together, and that the form $a/0$ indicates that *the problem is impossible*. That our readers may catch the spirit and meaning of his article, we have invited Dr. Lilley to give some elaboration to his views in a short article for the MONTHLY to be published in a future number. Although we do not approve some of the positions which the author has taken, still we regard the treatise on the whole as one of decided merit. The book has evidently been made for the class room and for actual use, and bears the marks of having been written by an experienced and practical teacher. We have only space to note further the demonstration for "Undetermined Coefficients," on page 419; "Pascal's Arithmetical Triangle," on page 442, which has published in the MONTHLY for December, 1894; and the many interesting notes on the subject of logarithms in the Appendix. J. M. C.

The following periodicals have been received: Journal de Mathématiques Élémentaires, (1er Juin 1897); American Journal of Mathematics, (April, 1897); The Mathematical Gazette, (February, 1897); L' Intermédiare des Mathématiciens, (Mai, 1897); Miscellaneous Notes and Queries, (May, 1897); The Kansas University Quarterly, (January, 1897); The Monist, (April, 1897); Bulletin of the American Mathematical Society, (May, (1897); The Educational Times, (May, 1897), Science, (No. for June 11, 1897); The Review of Reviews, (June, 1897), The Cosmopolitan, (June, 1897); The Arena, (June, 1897).



James Fyler
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